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we may take either $y_r = -a$, whence $x = (b+1)/10$; or $y_r = b$, $x = (-a+1)/10$. Thus solutions go in pairs, the x of either solution being obtained by adding 1 to the y_r of the other and dividing by 10. There are, then, always an even number of solutions, and always at least two, viz., those where $y_r = -1$, $x = 10^{r-1}(n+1)$; $y_r = 10^r(n+1) - 1$, $x = 0$.

Finally it may be noted that the more general equation

$$ax^r - bxy + y - c = 0$$

may be solved by similar methods. Here we should have to pick out such divisors of $b^r c - a$ as are $\equiv \pm 1 \pmod{b}$. It may be interesting to apply the method to an example. Let us solve

$$x^4 - 10xy - 22 + y = 0.$$

We have to solve

$$(10x - 1)y_4 = -219999 = -3 \times 13 \times 5641$$

(5641 prime). The values of y_4 with the corresponding solutions of our equation are:

$$y_4 = -5641, \quad x = 4, \quad y = 6; \quad y_4 = -1, \quad x = 22000 \quad (y \text{ a very large number});$$

$$y_4 = 39, \quad x = -564, \quad y = -17937434; \quad y_4 = 219999, \quad x = 0, \quad y = 22.$$

247 (Number Theory) [June, 1916]. Proposed by NORMAN ANNING, Chilliwack, B. C.

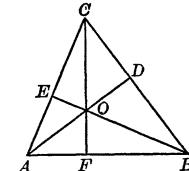
To dissect the triangle whose sides are 52, 56, 60 into three Heronian triangles by lines drawn from the vertices to a point within.

The word Heronian is used in the sense of the German Heronische (Wertheim, *Anfangsgründe d. Zahlenlehre*, p. 140) to describe a triangle whose sides and area are integral.

SOLUTION BY FRANK IRWIN, University of California.

The orthocenter, O , may be taken as the required point. Let ABC be the triangle with $a = 60$, $b = 52$, $c = 56$; and let the feet of the perpendiculars from A , B , C on the opposite sides be D , E , F , respectively. Then the various lines in the figure, calculated as indicated below, are: $BD = 168/5$, $DC = 132/5$, $CE = 396/13$, $EA = 280/13$, $AF = 20$, $FB = 36$; $AO = 25$, $BO = 39$, $CO = 33$. Finally, area $BOC = 594$, area $COA = 330$, and area $AOB = 420$; so that the sides and areas of these three triangles are integral, as asserted.

The explanation of these facts depends on the following proposition: If the sides and area of the triangle ABC are rational, the same is true of the triangles BOC , COA , AOB . (Then by multiplying the dimensions of the figure by a suitable integer everything can be made integral.) For the three altitudes are rational, as also the radius r of the inscribed circle (since $rs = \text{area}$). Thus $\tan A/2$ is rational, and so, then, are $\cos^2 A/2$ and $\cos A$. Therefore, $AF = b \cos A$ is rational, and similarly, FB , BD , etc. Then the triangle AOF is rational (that is, has rational sides), since one of its sides, AF , is rational, and it is similar to the rational triangle ABD .



2678 [February, 1918]. Problem proposed by C. F. GUMMER, Queen's University, Canada.

Find necessary and sufficient conditions that the roots of the equation $x^{n+1} + a_1x^n + a_2x^{n-1} + \cdots + a_{n+1} = 0$ may be all real and separated by the roots of $x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0$.

SOLUTION BY THE PROPOSER.

Consider the equations

$$(1) \quad f(x) = x^{n+1} + a_1x^n + \cdots = 0,$$

$$(2) \quad g(x) = x^n + b_1x^{n-1} + \cdots = 0,$$

$$(3) \quad R_1(x) = c_0x^{n-p} + c_1x^{n-p-1} + \cdots = 0,$$

$$(4) \quad R_2(x) = d_0x^{n-p-q} + d_1x^{n-p-q-1} + \cdots = 0,$$

where $R_1(x)$ is the remainder with sign changed on dividing $f(x)$ by $g(x)$, $R_2(x)$ has the same relation to $g(x)$ and $R_1(x)$, etc.

That the roots of (1) may be real and separated by those of (2), it is necessary and sufficient that

- (a) the roots of $g(x)$ shall be real and distinct (such as $\beta_1 < \beta_2 < \dots < \beta_n$) and
- (b) $f(-\infty), f(\beta_1), \dots, f(\beta_n), f(\infty)$ shall have alternate signs.

Now $f(\beta_i) = -R_i(\beta_i)$, and $R_i(x)$ cannot change sign more than $n - 1$ times. Hence, (a) and (b) are equivalent to the conditions that c_0 shall be positive, and that the roots of (2) are all real and separated by those of (3) (implying that $p = 1$). These are in turn equivalent to the conditions that c_0, d_0 shall be positive, $p = q = 1$, and the roots of (3) real and separated by those of (4), and so on.

Hence, a set of necessary and sufficient conditions is

$$p = q = \dots = 1; c_0, d_0, \dots, \text{all positive.}$$

It may be shown that these are equivalent to the conditions

$$\begin{vmatrix} 1 & b_1 & b_2 & b_3 & b_4 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 1 & a_1 & a_2 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & 1 & b_1 & b_2 \end{vmatrix}, \quad \dots$$

all positive. For the calculation of the remainders see papers by E. B. Van Vleck (*Annals of Mathematics*, 1899, pp. 1-13) and A. J. Pell and R. L. Gordon (*Annals of Mathematics*, June, 1917, pp. 188-193).

2701 [May, 1918]. Proposed by E. H. WORTHINGTON, University of Pennsylvania.

Find the sum of the infinite series

$$\frac{1}{5} + \frac{1 \cdot 2}{5 \cdot 7} r + \frac{1 \cdot 2 \cdot 3}{5 \cdot 7 \cdot 9} r^2 + \dots + \frac{n!}{5 \cdot 7 \cdot 9 \dots (2n+3)} r^{n-1} + \dots$$

Verify your result for $r = 0$ and $r = 1$.

SOLUTION BY A. M. HARDING, University of Arkansas.

It can be easily shown that the following equation holds for all integral values of $n \geq 0$:

$$\int_0^1 t^{1/2} (1-t)^{n+1} dt = \frac{2(n+1)}{2n+5} \int_0^1 t^{1/2} (1-t)^n dt.$$

Multiplying both sides by x^n and substituting $n = 0, 1, 2, \dots (n-1)$, gives

$$\begin{aligned} \frac{3}{4} \int_0^1 t^{1/2} (1-t) dt &= \frac{1}{5}, \\ \frac{3}{4} \int_0^1 t^{1/2} (1-t)^2 x dt &= \frac{1 \cdot 2}{5 \cdot 7} 2x, \\ \frac{3}{4} \int_0^1 t^{1/2} (1-t)^3 x^2 dt &= \frac{1 \cdot 2 \cdot 3}{5 \cdot 7 \cdot 9} (2x)^2, \\ &\dots \\ \frac{3}{4} \int_0^1 t^{1/2} (1-t)^n x^{n-1} dt &= \frac{n!}{5 \cdot 7 \cdot 9 \dots (2n+3)} (2x)^{n-1}. \\ &\dots \end{aligned}$$

Letting $x = r/2$ and adding, we have

$$\begin{aligned} \frac{3}{4} \int_0^1 t^{1/2} (1-t) [1 + (1-t)x + \dots + (1-t)^{n-1} x^{n-1} + \dots] dt \\ &= \frac{1}{5} + \frac{1 \cdot 2}{5 \cdot 7} r + \dots + \frac{n!}{5 \cdot 7 \cdot 9 \dots (2n+3)} r^{n-1} + \dots \end{aligned}$$

The series in the right member of this equation converges if $r < 2$. In this case, the series under the integral converges uniformly to the sum $\frac{1}{1 - (1-t)r/2}$. Hence, the sum of the given series is given by